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BURKHARDT QUARTIC, BARTH SEXTIC, AND THE ICOSAHDREDON

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ABSTRACT. We study two rational Fano threefolds with an action of the icosahedral group \mathfrak{A}_5 . The first one is the famous Burkhardt quartic threefold, and the second one is the double cover of the projective space branched in the Barth sextic surface. We prove that both of them are \mathfrak{A}_5 -Fano varieties that are \mathfrak{A}_5 -birationally superrigid. This gives two new embeddings of the group \mathfrak{A}_5 into the space Cremona group.

1. INTRODUCTION

One of the interesting subjects in three-dimensional birational geometry is studying G -Fano threefolds (see [Pr13]), where G is a finite group. These are Fano threefolds with terminal singularities such that the rank of their G -invariant class group equals 1. Finding enough details about their G -equivariant (biregular) geometry often leads to conclusions about the absence of G -equivariant birational maps between them, see e.g. [CS14] and [CS15]. It is a usual case that on this way one encounters really beautiful geometric constructions arising from large groups of symmetries of the corresponding varieties.

The icosahedral group \mathfrak{A}_5 appears as a group of symmetries for a remarkably large class of Fano threefolds, and many of them are \mathfrak{A}_5 -Fano threefolds, see e.g. [CS15], [CPS16], and [PS16]. In this paper we study two rational Fano threefolds with an action of the group \mathfrak{A}_5 . Both of these threefolds are singular and have only nodes as singularities; moreover, they are “extremal” in some sense, which makes their geometry more interesting. We prove that the ranks of their \mathfrak{A}_5 -invariant class groups equal 1, so that both of them are \mathfrak{A}_5 -Fano threefolds. We use this to prove that they are \mathfrak{A}_5 -birationally superrigid, see [CS15, §3.1] for a definition.

Consider the projective space \mathbb{P}^5 with homogeneous coordinates x_0, \dots, x_5 . Denote by $\sigma_k(x_0, \dots, x_5)$ the k -th elementary symmetric polynomial in x_0, \dots, x_5 . The *Burkhardt quartic* is defined in \mathbb{P}^5 by equations

$$\sigma_1(x_0, \dots, x_5) = \sigma_4(x_0, \dots, x_5) = 0.$$

This quartic threefold was first described by Burkhardt in [Bu91]. It has 45 nodes, which is actually the largest possible number of isolated singularities for quartic threefolds, see [JS-BV90] (cf. [Pr17]).

The Burkhardt quartic is known to be rational (see [To35]), and its automorphism group is isomorphic to $\mathrm{PSp}_4(\mathbf{F}_3)$, see [Co06, §6] and [JS-BV90]. Let the group \mathfrak{S}_6 act on \mathbb{P}^5 by permutations of homogeneous coordinates x_0, \dots, x_5 . Then \mathfrak{S}_6 preserves the Burkhardt quartic.

Let us refer to a subgroup \mathfrak{A}_5 in \mathfrak{S}_6 that fixes one of the homogeneous coordinates x_0, \dots, x_5 as a *standard* subgroup \mathfrak{A}_5 , and to a subgroup \mathfrak{A}_5 of \mathfrak{S}_6 that does not fix any of these coordinates as a *non-standard* subgroup \mathfrak{A}_5 . Our first result is

Theorem 1.1. *Let G be a standard subgroup \mathfrak{A}_5 in \mathfrak{S}_6 . The Burkhardt quartic is a rational G -Fano threefold that is G -birationally superrigid.*

It should be pointed out that if G' is a non-standard subgroup \mathfrak{A}_5 in \mathfrak{S}_6 , then the Burkhardt quartic is *not* a G' -Fano threefold (see Remark 2.12 below).

Now consider the projective space \mathbb{P}^3 that is a projectivization of a non-trivial reducible four-dimensional representation of the group \mathfrak{A}_5 . There exists a unique \mathfrak{A}_5 -invariant sextic surface \mathcal{B} with 65 nodes in \mathbb{P}^3 , which is actually the largest possible number of nodes for a sextic surface, see [JR97]. This surface was discovered by Barth in [Ba96] and is usually called the *Barth sextic*. Our second result is

Theorem 1.2. *The double cover of \mathbb{P}^3 branched over the surface \mathcal{B} is a rational \mathfrak{A}_5 -Fano three-fold that is \mathfrak{A}_5 -birationally superrigid.*

Theorems 1.1 and 1.2 give two embeddings of the group \mathfrak{A}_5 into the *space Cremona group* that are not conjugate to each other. Moreover, these embeddings are also not conjugate to any of the four embeddings described in [CS15, Remark 1.2.1], [CS15, Example 1.3.9], [CS15, Theorem 1.4.1], and [CPS16, Theorem 4.2].

Notation and conventions. All varieties are defined over the field \mathbb{C} of complex numbers. By $\text{Cl}(X)$ we denote the group of linear equivalence classes of Weil divisors on a variety X . If a variety V is acted on by a finite group G , and Z is a subvariety of V , we will sometimes abuse terminology and refer to the union of the images $g(Z)$, $g \in G$, as the G -orbit of Z . By a node we always mean an isolated singularity that is locally isomorphic to the singularity of a cone over a smooth quadric of the appropriate dimension.

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2. BURKHARDT QUARTIC

In this section we prove Theorem 1.1. Recall that the Burkhardt quartic X is given in the projective space \mathbb{P}^5 with homogeneous coordinates x_0, \dots, x_5 by equations

$$\sigma_1(x_0, \dots, x_5) = \sigma_4(x_0, \dots, x_5) = 0,$$

where $\sigma_k(x_0, \dots, x_5)$ is the k -th elementary symmetric polynomial in x_0, \dots, x_5 . The quartic X is rational, and its automorphism group is isomorphic to $\text{PSp}_4(\mathbb{F}_3)$. A subgroup $\mathfrak{S}_6 \subset \text{PSp}_4(\mathbb{F}_3)$ that acts by permutations of the homogeneous coordinates x_0, \dots, x_5 preserves X . The Burkhardt quartic has 45 singularities that form two \mathfrak{S}_6 -orbits: one is the \mathfrak{S}_6 -orbit of length 30 of the point

$$[1 : 1 : \omega : \omega : \omega^2 : \omega^2],$$

and the other is the \mathfrak{S}_6 -orbit of length 15 of the point

$$[1 : -1 : 0 : 0 : 0 : 0].$$

Remark 2.1 (cf. [C⁺, p. 26]). Up to conjugation, the group $\text{PSp}_4(\mathbb{F}_3)$ contains a unique subgroup isomorphic to \mathfrak{S}_6 , and two subgroups isomorphic to \mathfrak{A}_5 . The latter agree with two non-conjugate embeddings of \mathfrak{A}_5 to $\mathfrak{S}_6 \subset \text{PSp}_4(\mathbb{F}_3)$.

Lemma 2.2. *One has $\text{rk Cl}(X)^{\mathfrak{S}_6} = 1$.*

Proof. It is well-known that the quotient $\mathbb{P}^5/\mathfrak{S}_6$ is isomorphic to the weighted projective space $\mathbb{P}(1, 2, 3, 4, 5, 6)$, so that

$$\mathbb{P}^4/\mathfrak{S}_6 \cong \mathbb{P}(2, 3, 4, 5, 6)$$

and $X/\mathfrak{S}_6 \cong \mathbb{P}(2, 3, 5, 6)$. In particular, one has $\text{rk Cl}(X)^{\mathfrak{S}_6} = \text{rk Cl}(X/\mathfrak{S}_6) = 1$. \square

Corollary 2.3. *One has $\mathrm{rk} \mathrm{Cl}(X)^{\mathrm{PSp}_4(\mathbf{F}_3)} = 1$.*

If one ignores the action of the automorphism group of X , the following result is known.

Lemma 2.4 ([Ka11, Theorem 1.1(iii)]). *One has $\mathrm{rk} \mathrm{Cl}(X) = 16$.*

We are going to find the ranks of the invariant parts of $\mathrm{Cl}(X)$ with respect to various groups. Consider the vector space

$$\mathrm{Cl}(X)_{\mathbb{C}} = \mathrm{Cl}(X) \otimes \mathbb{C}$$

as a representation of the group $\mathrm{PSp}_4(\mathbf{F}_3)$. Then $\mathrm{Cl}(X)_{\mathbb{C}}$ contains a trivial subrepresentation \mathbb{K} corresponding to the canonical class, and one has $\mathrm{Cl}(X)_{\mathbb{C}} \cong \mathbb{K} \oplus \mathbb{W}$. We know from Lemma 2.4 that \mathbb{W} is a 15-dimensional representation of $\mathrm{PSp}_4(\mathbf{F}_3)$. Obviously, \mathbb{W} is defined over the field \mathbb{Q} .

Lemma 2.5. *The complex $\mathrm{PSp}_4(\mathbf{F}_3)$ -representation \mathbb{W} is irreducible.*

Proof. Recall that all non-trivial irreducible representations of $\mathrm{PSp}_4(\mathbf{F}_3)$ of dimension less than 15 are two 5-dimensional representations χ_2 and χ_3 that are not defined over \mathbb{Q} , one 6-dimensional representation χ_4 , and two 10-dimensional representations χ_5 and χ_6 that are not defined over \mathbb{Q} (see [C⁺, p. 27]).

Suppose that \mathbb{W} is reducible. Note that \mathbb{W} does not have trivial subrepresentations by Corollary 2.3. Therefore, \mathbb{W} splits as a sum of representations of $\mathrm{PSp}_4(\mathbf{F}_3)$ isomorphic to χ_2 , χ_3 , χ_4 , χ_5 , or χ_6 . However, keeping in mind that $\dim(\mathbb{W}) = 15$ we see that there are no summands isomorphic to χ_4 . Moreover, since χ_5 and χ_6 are not defined over \mathbb{Q} , they either appear as summands in \mathbb{W} simultaneously, or do not appear at all. The same holds for χ_2 and χ_3 . An obtained contradiction completes the proof of the lemma. \square

By Lemma 2.5, the $\mathrm{PSp}_4(\mathbf{F}_3)$ -representation \mathbb{W} is isomorphic to one of the two 15-dimensional irreducible representations of $\mathrm{PSp}_4(\mathbf{F}_3)$ (see [C⁺, p. 27]). In the notation of [C⁺, p. 27] these are χ_7 and χ_8 . In fact, we have $\mathbb{W} \cong \chi_7$. This follows from Lemma 2.2 and (the first part of) the following result.

Lemma 2.6. *The following assertions hold:*

- (i) *the \mathfrak{S}_6 -representation $\chi_7|_{\mathfrak{S}_6}$ does not contain trivial subrepresentations, while the \mathfrak{S}_6 -representation $\chi_8|_{\mathfrak{S}_6}$ does;*
- (ii) *for one of the two non-conjugate embeddings of \mathfrak{A}_5 to $\mathrm{PSp}_4(\mathbf{F}_3)$, the \mathfrak{A}_5 -representation $\chi_7|_{\mathfrak{A}_5}$ does not contain trivial subrepresentations, while for the other embedding it contains a unique trivial subrepresentation.*

Proof. Both assertions are obtained by direct computations. We used the GAP software [GAP] to perform them. \square

Note that Lemma 2.6(ii) implies that the invariant part of $\mathrm{Cl}(X)$ with respect to one of the two non-conjugate actions of \mathfrak{A}_5 on X has rank 1.

Later we will need the following elementary result.

Lemma 2.7. *Let Y be a normal variety acted on by a finite group G . Suppose that there exist Weil divisors Π_1, \dots, Π_r on Y such that they generate the \mathbb{Q} -vector space*

$$\mathrm{Cl}(Y)_{\mathbb{Q}} = \mathrm{Cl}(Y) \otimes \mathbb{Q},$$

and Π_1, \dots, Π_r form one G -orbit. Then the \mathbb{Q} -vector space

$$\mathrm{Cl}(Y)_{\mathbb{Q}}^G = \mathrm{Cl}(Y)^G \otimes \mathbb{Q}$$

is one-dimensional.

Proof. Let D be a G -invariant Weil divisor on Y . By assumption, we have

$$D \sim_{\mathbb{Q}} \sum_{i=1}^r a_i \Pi_i$$

for some rational numbers a_1, \dots, a_r . Put $\mathcal{P} = \Pi_1 + \dots + \Pi_r$. Then

$$|G|D \sim \sum_{g \in G} g(D) \sim_{\mathbb{Q}} \sum_{i=1}^r a_i \sum_{g \in G} g(\Pi_i) = \sum_{i=1}^r a_i \frac{|G|}{r} \mathcal{P} = \frac{|G|}{r} \left(\sum_{i=1}^r a_i \right) \mathcal{P}.$$

In particular, we see that the \mathbb{Q} -vector space $\text{Cl}(Y)_{\mathbb{Q}}^G$ is generated by \mathcal{P} . \square

Corollary 2.8. *Let Y be a nodal Fano threefold acted on by a finite group G , and let i_Y be the largest positive integer such that $-K_Y \sim i_Y H$, where H is an ample Cartier divisor on Y . Suppose that there exist Weil divisors Π_1, \dots, Π_r on Y such that they generate the \mathbb{Q} -vector space*

$$\text{Cl}(Y)_{\mathbb{Q}} = \text{Cl}(Y) \otimes \mathbb{Q},$$

and Π_1, \dots, Π_r form one G -orbit. Then $\text{Cl}(Y)^G = \mathbb{Z} \cdot H$.

Proof. By Lemma 2.7, the \mathbb{Q} -vector space $\text{Cl}(Y)_{\mathbb{Q}}^G$ is one-dimensional. Since any singular point O of Y is a node, we know that any Weil divisor that is \mathbb{Q} -Cartier in a neighborhood of O is actually Cartier in a neighborhood of O . In particular, every G -invariant Weil divisor on Y is a Cartier divisor. Since the Picard group of Y has no torsion (this holds even for Fano varieties with log terminal singularities, see e.g. [IP99, Proposition 2.1.2]), this implies that $\text{rk Cl}(Y)^G = 1$, and the assertion follows. \square

Now we are ready to start proving Theorem 1.1. Recall that a subgroup \mathfrak{A}_5 in \mathfrak{S}_6 that fixes one of the homogeneous coordinates x_0, \dots, x_5 is called a *standard* subgroup \mathfrak{A}_5 , and a subgroup \mathfrak{A}_5 of \mathfrak{S}_6 that does not fix any of these homogeneous coordinates is called a *non-standard* subgroup \mathfrak{A}_5 . By Remark 2.1 subgroups of these two kinds represent two conjugacy classes of subgroups isomorphic to \mathfrak{A}_5 in $\text{PSp}_4(\mathbb{F}_3)$.

Let G be a standard subgroup \mathfrak{A}_5 in \mathfrak{S}_6 that fixes the homogeneous coordinate x_5 . We are going to prove the following result.

Proposition 2.9. *The group $\text{Cl}(X)^G$ is generated by $-K_X$.*

As a consequence of Proposition 2.9 we can find the rank of the Weil divisor class group invariant under any given group that contains a standard subgroup \mathfrak{A}_5 . In particular, for a subgroup $\mathfrak{A}_6 \subset \mathfrak{S}_6$ we get

Corollary 2.10. *One has $\text{rk Cl}(X)^{\mathfrak{A}_6} = 1$.*

Remark 2.11. The assertion of Corollary 2.10 was used in the proof of [CS14, Theorem 1.20]. Moreover, the authors of [CS14] gave a brief sketch of a proof of this assertion, but this proof was actually incorrect. Indeed, contrary to what was claimed in the proof of [CS14, Theorem 1.20], the quotient of X by a subgroup $\mathfrak{S}_4 \subset \mathfrak{A}_6$ is not isomorphic to the weighted projective space $\mathbb{P}(1, 2, 2, 3)$. Our proof of Corollary 2.10 fixes this gap and thus recovers the proof of [CS14, Theorem 1.20].

Remark 2.12. Let G' be a non-standard subgroup $\mathfrak{A}_5 \subset \mathfrak{S}_6$. Then Proposition 2.9 and Lemma 2.6(ii) imply that $\text{rk Cl}(X)^{G'} = 2$.

Now we derive Theorem 1.1 from Proposition 2.9.

Proof of Theorem 1.1. The proof is standard, see [CS14, Theorem 1.20] and [Me04], but we include it for the reader's convenience.

Suppose that X is not G -birationally superrigid. Since the divisor $-K_X$ generates the group $\text{Cl}(X)^G$ by Proposition 2.9, it follows from [CS15, Corollary 3.3.3] that there exist a positive integer n and a G -invariant linear subsystem \mathcal{M} of the linear system $|-nK_X|$ such that \mathcal{M} does not have fixed components, and the log pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical. Choose two general surfaces M_1 and M_2 in the linear system \mathcal{M} , and denote by H a general hyperplane section of X . Then $H \sim -K_X$.

Suppose that there is an irreducible curve $C \subset X$ such that the log pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical along C . Denote by Z the G -orbit of the curve C . Put $d = H \cdot Z$ and $m = \text{mult}_C(M_1) = \text{mult}_C(M_2)$. Then $m > n$, so that

$$4n^2 = M_1 \cdot M_2 \cdot H \geq dm^2 > dn^2,$$

which gives $d \leq 3$. In particular, one has $Z = C$. Since \mathbb{P}^4 does not contain G -invariant lines and planes, we see that Z is a twisted cubic. Moreover, the curve Z is contained in the smooth locus of X , because the possible lengths of G -orbits in Z are 12, 20, 30, 60, and there are no G -orbits of such lengths consisting of singular points of X .

Let $f: W \rightarrow X$ be the blow up along the curve Z . Denote by F the f -exceptional divisor. Denote by \widetilde{M}_1 and \widetilde{M}_2 the proper transforms of the surfaces M_1 and M_2 on the threefold W , respectively. We then get

$$\widetilde{M}_1 \sim \widetilde{M}_2 \sim f^*(nH) - mF.$$

Moreover, the divisor $f^*(2H) - F$ is nef, because Z is a scheme-theoretic intersection of quadrics. Keeping in mind that $F^3 = -K_X \cdot Z - 2 = 1$ and $H \cdot Z = 3$, we obtain

$$0 \leq (f^*(2H) - F) \cdot \widetilde{M}_1 \cdot \widetilde{M}_2 = (f^*(2H) - F) \left(f^*(nH) - mF \right)^2 = 8n^2 - 6nm - 5m^2 < 0$$

because $m > n$. This is a contradiction.

Thus, the log pair $(X, \lambda\mathcal{M})$ is canonical outside of finitely many points of X . Take any point $P \in X$ such that the singularities of the log pair $(X, \lambda\mathcal{M})$ are not canonical at the point P . Suppose that P is a smooth point of X , and let H_P be a general hyperplane section of X passing through P . Then

$$4m = M_1 \cdot M_2 \cdot H_P \geq \text{mult}_P(M_1 \cdot M_2) > 4n^2$$

by [Co00, Corollary 3.4]. The obtained contradiction shows that X is singular at P .

Let Σ be the G -orbit of the point P . Then there is a subset $\Gamma \subset \Sigma$ such that $|\Gamma| = 4$, and the set Γ is not contained in any plane in \mathbb{P}^4 . Let $g: U \rightarrow X$ be a blow up of Γ , and let E_1, E_2, E_3 , and E_4 be exceptional divisors of g . Denote by \overline{M}_1 and \overline{M}_2 the proper transforms of the surfaces M_1 and M_2 on the threefold U , respectively. We then get

$$\overline{M}_1 \sim \overline{M}_2 \sim g^*(nH) - \delta \sum_{i=1}^4 E_i.$$

for some positive integer δ . Moreover, it follows from [Co00, Theorem 3.10] that $\delta > n$. On the other hand, the divisor $g^*(2H) - E_1 - E_2 - E_3 - E_4$ is nef, because the points of Γ are not coplanar. In particular, we have

$$0 \leq \left(g^*(2H) - \sum_{i=1}^4 E_i \right) \cdot \overline{M}_1 \cdot \overline{M}_2 = \left(g^*(2H) - \sum_{i=1}^4 E_i \right) \left(g^*(nH) - \delta \sum_{i=1}^4 E_i \right)^2 = 8n^2 - 8\delta^2.$$

This is impossible, since $\delta > n$. □

In the rest of this section we give a proof of Proposition 2.9. Recall from [Hu96, §5.2.1] that the Burkhardt quartic X contains forty j -planes, that are planes passing through nine singular points of X . Let us describe them. For any triple of indices $0 \leq i_1 < i_2 < i_3 \leq 5$, we denote by $\Pi_{i_1 i_2 i_3}^+$ the plane given in \mathbb{P}^5 by equations

$$x_{i_2} = \omega x_{i_1}, \quad x_{i_3} = \omega^2 x_{i_1}, \quad \sigma_1(x_0, \dots, x_5) = 0,$$

where ω is a primitive cubic root of 1, and we denote by $\Pi_{i_1 i_2 i_3}^-$ the plane given in \mathbb{P}^5 by equations

$$x_{i_3} = \omega x_{i_1}, \quad x_{i_2} = \omega^2 x_{i_1}, \quad \sigma_1(x_0, \dots, x_5) = 0.$$

The 40 planes $\Pi_{i_1 i_2 i_3}^\pm$ form one \mathfrak{S}_6 -orbit. On the other hand, these 40 planes split into two G -orbits of length 20, one containing the planes $\Pi_{i_1 i_2 i_3}^\pm$ for $0 \leq i_1 < i_2 < i_3 \leq 4$, and the other containing $\Pi_{i_1 i_2 5}^\pm$ for $0 \leq i_1 < i_2 \leq 4$.

Consider the following intersection form on the lattice $\text{Cl}(X)$. Choose a general (smooth) hyperplane section H of X ; in particular, we assume that H does not pass through the singular points of X . Given two Weil divisors D_1 and D_2 on X , we restrict them to H (which makes sense since in appropriately chosen neighborhoods of the intersection of their supports with H they are actually Cartier divisors), and define $D_1 \bullet D_2$ as the intersection of the resulting curves on H .

Lemma 2.13. *Choose two triples of indices*

$$0 \leq i_1 < i_2 < i_3 \leq 5, \quad 0 \leq j_1 < j_2 < j_3 \leq 5.$$

Let c be the cardinality of the set $\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\}$. If $c = 2$, put $\delta = 1$ provided that one can choose indices $1 \leq a < b \leq 3$ and $1 \leq a' < b' \leq 3$ such that $i_a = j_{a'}$, $i_b = j_{b'}$, and $b - a = b' - a'$; otherwise put $\delta = 0$. The following assertions hold:

- (o) *if $c = 0$, then $\Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{j_1 j_2 j_3}^+ = \Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{j_1 j_2 j_3}^- = \Pi_{i_1 i_2 i_3}^- \bullet \Pi_{j_1 j_2 j_3}^- = 1$;*
- (i) *if $c = 1$, then $\Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{j_1 j_2 j_3}^+ = \Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{j_1 j_2 j_3}^- = \Pi_{i_1 i_2 i_3}^- \bullet \Pi_{j_1 j_2 j_3}^- = 0$;*
- (ii) *if $c = 2$, then $\Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{j_1 j_2 j_3}^+ = \Pi_{i_1 i_2 i_3}^- \bullet \Pi_{j_1 j_2 j_3}^- = \delta$, and $\Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{j_1 j_2 j_3}^- = 1 - \delta$;*
- (iii) *one has $\Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{i_1 i_2 i_3}^+ = \Pi_{i_1 i_2 i_3}^- \bullet \Pi_{i_1 i_2 i_3}^- = -2$, and $\Pi_{i_1 i_2 i_3}^+ \bullet \Pi_{i_1 i_2 i_3}^- = 1$.*

Proof. The self-intersection number -2 corresponds to the self-intersection of a smooth rational curve on a $K3$ surface. The cases with intersection number 1 correspond to pairs of planes that meet along a line, and the cases with intersection number 0 correspond to pairs of planes that meet at a point. \square

Corollary 2.14. *The 20×20 matrix of intersection numbers of the planes $\Pi_{i_1 i_2 i_3}^\pm$, where $0 \leq i_1 < i_2 < i_3 \leq 4$, has rank 16. Similarly, the 20×20 matrix of intersection numbers of the planes $\Pi_{i_1 i_2 5}^\pm$, where $0 \leq i_1 < i_2 \leq 4$, also has rank 16.*

Proof. Straightforward computation. \square

Lemma 2.4 and Corollary 2.14 imply the following result.

Corollary 2.15. *The classes of the 20 planes $\Pi_{i_1 i_2 i_3}^\pm$, where $0 \leq i_1 < i_2 < i_3 \leq 4$, generate the \mathbb{Q} -vector space*

$$\text{Cl}(X)_{\mathbb{Q}} = \text{Cl}(X) \otimes \mathbb{Q}.$$

Similarly, the classes of the 20 planes $\Pi_{i_1 i_2 5}^\pm$, where $0 \leq i_1 < i_2 \leq 4$, also generate the \mathbb{Q} -vector space $\text{Cl}(X)_{\mathbb{Q}}$.

By Corollary 2.8, the assertion of Proposition 2.9 follows from Corollary 2.15. This completes the proof of Theorem 1.1.

3. BARTH SEXTIC DOUBLE SOLID

Let \mathbb{I} be the trivial representation of the group \mathfrak{A}_5 , and let \mathbb{V} be one of its two three-dimensional irreducible representations (see e.g. [C⁺, p. 2]). Put $\mathbb{P}^3 = \mathbb{P}(\mathbb{I} \oplus \mathbb{V})$. By [Ba96], there exists a unique \mathfrak{A}_5 -invariant sextic surface \mathcal{B} in \mathbb{P}^3 with 65 isolated singular points. Moreover, the surface \mathcal{B} is given in appropriate homogeneous coordinates x_0, x_1, x_2, x_3 by equation

$$(3.1) \quad 4(\tau^2 x_0^2 - x_1^2)(\tau^2 x_1^2 - x_2^2)(\tau^2 x_2^2 - x_0^2) - (1 + 2\tau)x_3^2(x_0^2 + x_1^2 + x_2^2 - x_3^2)^2 = 0,$$

where $\tau = \frac{1+\sqrt{5}}{2}$, and it has only nodes as singularities.

Recall from [Ba96, §1] that the group \mathfrak{A}_5 acting on \mathbb{P}^3 so that the sextic \mathcal{B} is \mathfrak{A}_5 -invariant can be thought of as the group of rotations of an icosahedron with 12 vertices

$$[\pm\tau : \pm 1 : 0 : 1], \quad [0 : \pm\tau : \pm 1 : 1], \quad [\pm 1 : 0 : \pm\tau : 1].$$

In particular, the group \mathfrak{A}_5 contains the transformation

$$(3.2) \quad x_0 \mapsto -x_0, \quad x_1 \mapsto -x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3$$

of order 2, the transformation

$$(3.3) \quad x_0 \mapsto x_1 \mapsto x_2 \mapsto x_0, \quad x_3 \mapsto x_3$$

of order 3, and the transformation

$$(3.4) \quad (x_0, x_1, x_2) \mapsto (x_0, x_1, x_2)M^T, \quad x_3 \mapsto x_3,$$

where M is the matrix

$$\begin{aligned} & \begin{pmatrix} \frac{\tau}{\sqrt{\tau+2}} & -\frac{1}{\sqrt{\tau+2}} & 0 \\ \frac{1}{\sqrt{\tau+2}} & \frac{\tau}{\sqrt{\tau+2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{2\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) \\ 0 & \sin\left(\frac{2\pi}{5}\right) & \cos\left(\frac{2\pi}{5}\right) \end{pmatrix} \begin{pmatrix} \frac{\tau}{\sqrt{\tau+2}} & -\frac{1}{\sqrt{\tau+2}} & 0 \\ \frac{1}{\sqrt{\tau+2}} & \frac{\tau}{\sqrt{\tau+2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \\ & = \begin{pmatrix} \frac{\sqrt{5}(3+\sqrt{5})}{2(5+\sqrt{5})} & \frac{\sqrt{5}}{5+\sqrt{5}} & \frac{1}{2} \\ \frac{\sqrt{5}}{5+\sqrt{5}} & \frac{1}{2} & \frac{-\sqrt{5}-1}{4} \\ -\frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \tau & \tau-1 & 1 \\ \tau-1 & 1 & -\tau \\ -1 & \tau & \tau-1 \end{pmatrix}. \end{aligned}$$

The latter is a transformation of order 5 which corresponds to a rotation around the axis through the vertices $[\tau : 1 : 0 : 1]$ and $[-\tau : -1 : 0 : 1]$ of the icosahedron by the angle $2\pi/5$.

Let Σ_{15} be the \mathfrak{A}_5 -orbit of the point $[1 : 0 : 0 : 0]$, let Σ_{30} be the \mathfrak{A}_5 -orbit of the point $[1 : 0 : 0 : 1]$, and let Σ_{20} be the \mathfrak{A}_5 -orbit of the point $[1 : 1 : 1 : 1]$. Then one has $|\Sigma_k| = k$, and one can check that the sextic surface \mathcal{B} is singular at the points of these three \mathfrak{A}_5 -orbits. Moreover, one has $\text{Sing}(\mathcal{B}) = \Sigma_{15} \cup \Sigma_{20} \cup \Sigma_{30}$, see [Ba96, §1].

Remark 3.1. Restricting the left hand side of (3.1) to the plane $x_3 = x_0 + x_1 + x_2$ we get an equation

$$-4(5\tau + 3) \left((\tau - 2)(x_0x_1^2 + x_1x_2^2 + x_2x_0^2) + (\tau - 3)x_0x_1x_2 - (x_0^2x_1 + x_1^2x_2 + x_2^2x_0) \right)^2 = 0.$$

Similarly, restricting the left hand side of (3.1) to the plane $x_3 = x_0 - x_1 - x_2$ we get an equation

$$-4(5\tau + 3) \left((2 - \tau)(x_0x_1^2 - x_1x_2^2 - x_2x_0^2) + (3 - \tau)x_0x_1x_2 - (x_0^2x_1 + x_1^2x_2 - x_2^2x_0) \right)^2 = 0.$$

Define the plane $\Xi_{(v_0, v_1, v_2)}$ in \mathbb{P}^3 by equation $x_3 = v_0x_0 + v_1x_1 + v_2x_2$, where $v = (v_0, v_1, v_2)$ is one of the following collections of coefficients:

$$(3.5) \quad \begin{aligned} &(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1), \\ &(1, -1, -1), (-1, 1, -1), (-1, -1, 1), (-1, -1, -1), \\ &(\tau - 1, \tau, 0), (1 - \tau, \tau, 0), (1 - \tau, -\tau, 0), (\tau - 1, -\tau, 0), (\tau, 0, 1 - \tau), (\tau, 0, \tau - 1), \\ &(-\tau, 0, \tau - 1), (-\tau, 0, 1 - \tau), (0, \tau - 1, \tau), (0, \tau - 1, -\tau), (0, 1 - \tau, -\tau), (0, 1 - \tau, \tau). \end{aligned}$$

There are 20 planes like this, and they form a single \mathfrak{A}_5 -orbit. Similarly, define the plane $\Theta_{(u_0, u_1, u_2)}$ in \mathbb{P}^3 by equation $u_0x_0 + u_1x_1 + u_2x_2 = 0$, where $u = (u_0, u_1, u_2)$ is one of the following collections of coefficients:

$$(\tau, 1, 0), (\tau, -1, 0), (0, \tau, 1), (0, \tau, -1), (1, 0, \tau), (-1, 0, \tau).$$

There are 6 planes like this, and they form a single \mathfrak{A}_5 -orbit.

Lemma 3.2. *A restriction of the sextic \mathcal{B} to each of the planes Ξ_v is a smooth cubic curve taken with multiplicity 2. A restriction of the sextic \mathcal{B} to each of the planes Θ_u is a union of a line taken with multiplicity 2 and an irreducible conic taken with multiplicity 2.*

Proof. It is enough to check the assertion for one of the planes Ξ_v and one of the planes Θ_u . The restriction of \mathcal{B} to the plane $\Xi_{1,1,1}$ is given by equation

$$((\tau - 2)(x_0x_1^2 + x_1x_2^2 + x_2x_0^2) + (\tau - 3)x_0x_1x_2 - (x_0^2x_1 + x_1^2x_2 + x_2^2x_0))^2 = 0,$$

see Remark 3.1. Similarly, the restriction of \mathcal{B} to the plane $\Theta_{(-1,0,\tau)}$ is given by equation

$$x_3^2(x_1^2 + (1 + \tau^2)x_2^2 - x_3^2) = 0.$$

□

Denote by Υ the plane in \mathbb{P}^3 that is given by $x_3 = 0$, so that $\Upsilon \cong \mathbb{P}(\mathbb{V})$. Recall that for every $k \in \{6, 10, 15\}$ there is a unique \mathfrak{A}_5 -orbit Ω_k of length k in Υ , and there is a unique \mathfrak{A}_5 -invariant curve \mathcal{L}_k in Υ that is a union of k lines, see e.g. [CS15, Lemma 5.3.1(i),(ii)].

Lemma 3.3. *Let ℓ be a line in Υ that is not an irreducible component of \mathcal{L}_6 . Suppose that $(\ell \cdot \mathcal{L}_6)_P \geq 2$ for every point $P \in \ell \cap \mathcal{L}_6$. Then ℓ is an irreducible component of \mathcal{L}_{10} .*

Proof. By [CS15, Theorem 6.1.2(i)] the singular points of the curve \mathcal{L}_6 are the points of Ω_{15} ; the multiplicity of \mathcal{L}_6 at each of these points equals 2. Therefore, the line ℓ must contain three points of Ω_{15} . On the other hand, all lines passing through pairs of points of Ω_{15} are irreducible components of the curves \mathcal{L}_6 , or \mathcal{L}_{10} , or \mathcal{L}_{15} ; this follows from polarity (see [CS15, Remark 5.3.2]) and the fact that the points of pairwise intersections of the irreducible components of \mathcal{L}_{15} are the points of the \mathfrak{A}_5 -orbits Ω_6 , Ω_{10} , and Ω_{15} , see [CS15, Theorem 6.1.2(xvi)]. This implies that ℓ is an irreducible component of either \mathcal{L}_6 , or \mathcal{L}_{10} , or \mathcal{L}_{15} . However, the first of these cases does not occur by assumption, and the third is excluded by [CS15, Theorem 6.1.2(xiv)]. □

Note that the intersection $\Xi_v \cap \Upsilon$ is an irreducible component of \mathcal{L}_{10} . Indeed, this intersection is a line in Υ whose \mathfrak{A}_5 -orbit has length k that divides 20, and moreover $k < 20$ because $\Xi_{(1,1,1)}$ and $\Xi_{(-1,-1,-1)}$ intersect Υ by the same line. Similarly, the intersection $\Theta_u \cap \Upsilon$ is an irreducible component of \mathcal{L}_6 , because the latter is the only \mathfrak{A}_5 -orbit in Υ that consists of at most 6 lines.

Proposition 3.4. *Suppose that Π is a plane in \mathbb{P}^3 such that the restriction $\mathcal{B}|_\Pi$ is a cubic curve taken with multiplicity 2. Then Π is one of the planes Ξ_v or Θ_u .*

Proof. Observe that $\Pi \neq \Upsilon$, because $\mathcal{B}|_{\Upsilon} = \mathcal{L}_6$ is a reduced curve. So we put $\ell = \Pi \cap \Upsilon$. By Lemma 3.3, the line ℓ is an irreducible component of either \mathcal{L}_6 or \mathcal{L}_{10} . Since \mathfrak{A}_5 permutes transitively the irreducible components of each of the curves \mathcal{L}_6 and \mathcal{L}_{10} , we may assume that ℓ is given either by $\tau x_0 + x_1 = x_3 = 0$, or by $x_0 + x_1 + x_2 = x_3 = 0$.

Suppose that ℓ is given by $\tau x_0 + x_1 = x_3 = 0$. Then Π is given by the equation $\tau x_0 + x_1 = \lambda x_3$ for some $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $\Pi = \Theta_{(\tau,1,0)}$. If $\lambda \neq 0$, then the restriction $\mathcal{B}|_{\Pi}$ is given by

$$0 = 4(\tau^2 x_0^2 - x_1^2)(\tau^2 x_1^2 - x_2^2)(\tau^2 x_2^2 - x_0^2) - \frac{1+2\tau}{\lambda^2}(\tau x_0 + x_1)^2 \left(x_0^2 + x_1^2 + x_2^2 - \frac{1}{\lambda^2}(\tau x_0 + x_1)^2 \right)^2.$$

This (possibly non-reduced) sextic curve contains the line $\tau x_0 + x_1 = 0$ with multiplicity 1, so that it cannot be a double cubic.

Now we suppose that ℓ is given by $x_0 + x_1 + x_2 = x_3 = 0$. Then the plane Π is given by the equation $x_0 + x_1 + x_2 = \lambda x_3$ for some $\lambda \in \mathbb{C}$. If $\lambda = 1$, then $\Pi = \Xi_{(1,1,1)}$. If $\lambda = -1$, then $\Pi = \Xi_{(-1,-1,-1)}$.

Suppose that $\lambda \neq 0$. Put $\mu = \frac{1}{\lambda}$. Then $\mu \neq 0$, and the restriction $\mathcal{B}|_{\Pi}$ is given by the equation $f(x_0, x_1, x_2) = 0$, where

$$f(x_0, x_1, x_2) = 4(\tau^2 x_0^2 - x_1^2)(\tau^2 x_1^2 - x_2^2)(\tau^2 x_2^2 - x_0^2) - (1+2\tau)\mu^2(x_0 + x_1 + x_2)^2 (x_0^2 + x_1^2 + x_2^2 - \mu^2(x_0 + x_1 + x_2)^2)^2.$$

We have to show that the polynomial $f(x_0, x_1, x_2)$ is not a square of a cubic polynomial unless $\mu = \pm 1$. To show this it is enough to prove the same assertion for the polynomial

$$f(1, x_1, -x_1) = -(8\tau + 4)x_1^6 - (4 + 8\tau)(\mu^2 - 3)x_1^4 + (4 + 8\tau)(\mu^2 - \tau)(\mu^2 + \tau - 1)x_1^2 - (1 + 2\tau)\mu^2(\mu + 1)^2(\mu - 1)^2.$$

This follows from the fact that if $\mu \neq \pm 1$, then both the constant term and the leading coefficient of $f(1, x_1, -x_1)$ are not zero, while the coefficients at x_1 and x_1^3 are both zero.

Thus, we see that $\lambda = 0$, so that Π is given by $x_0 + x_1 + x_2 = 0$. Expressing $x_0 = -x_1 - x_2$, we see that the restriction $\mathcal{B}|_{\Pi}$ is given by the equation $g(x_1, x_2, x_3) = 0$, where

$$g(x_1, x_2, x_3) = 4(\tau^2(x_1 + x_2)^2 - x_1^2)(\tau^2 x_1^2 - x_2^2)(\tau^2 x_2^2 - (x_1 + x_2)^2) - (1+2\tau)x_3^2((x_1 + x_2)^2 + x_1^2 + x_2^2 - x_3^2)^2.$$

We have to show that the polynomial $g(x_1, x_2, x_3)$ is not a square of a cubic polynomial. To show this it is enough to prove the same assertion for the polynomial

$$g(x_1, -x_1, 1) = -(1+2\tau)(4x_1^6 + 4x_1^4 - 4x_1^2 + 1).$$

This polynomial is not a square of a cubic polynomial, because both its constant term and the leading coefficient are not zero, while the coefficients at x_1 and x_1^3 are both zero. \square

Let $\pi: X \rightarrow \mathbb{P}^3$ be a double cover branched over the sextic \mathcal{B} . The equation of X can be written in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$ with weighted homogeneous coordinates x_0, \dots, x_3 , and w as

$$(3.6) \quad w^2 + 4l_1 l_2 l_3 l_4 l_5 l_6 - q_3^2 = 0,$$

where $l_1 = \tau x_0 - x_1$, $l_2 = \tau x_1 - x_2$, $l_3 = \tau x_2 - x_0$, $l_4 = \tau x_0 + x_1$, $l_5 = \tau x_1 + x_2$, $l_6 = \tau x_2 + x_0$, and

$$q_3 = \sqrt{1+2\tau}x_3(x_0^2 + x_1^2 + x_2^2 - x_3^2).$$

The class group of the threefold X was described by Endrass.

Lemma 3.5 ([En99, Example 3.7]). *One has $\text{rk Cl}(X) = 14$.*

Proposition 3.6. *The threefold X is rational.*

Proof. Making a change of coordinates $w = 2yl_1l_2 + q_3$ we see that there is a birational map $\varphi: X \dashrightarrow Y$ to a quartic threefold Y given in the projective space \mathbb{P}^4 with homogeneous coordinates x_0, \dots, x_3, y by equation $y^2l_1l_2 + yq_3 + l_3l_4l_5l_6 = 0$. The map φ is given by the formula

$$[x_0 : x_1 : x_2 : x_3 : w] \mapsto [2l_1l_2x_0 : 2l_1l_2x_1 : 2l_1l_2x_2 : 2l_1l_2x_3 : w - q_3].$$

The inverse birational map $\psi: Y \dashrightarrow X$ is given by the Stein factorization of the linear projection from the point $[0 : 0 : 0 : 0 : 1]$ in \mathbb{P}^4 , so that ψ is defined by the formula

$$[x_0 : x_1 : x_2 : x_3 : y] \mapsto [x_0 : x_1 : x_2 : x_3 : 2l_1l_2y + q_3].$$

The quartic Y contains a plane Π given by equations $y = l_4 = 0$. The projection $\sigma: Y \dashrightarrow \mathbb{P}^1$ from Π is given by $[\lambda : \mu] = [y : l_4]$, where λ and μ are homogeneous coordinates on \mathbb{P}^1 .

Putting $\lambda = \frac{y}{l_4}$ and $\mu = 1$, we see that the general fiber of σ (in a scheme sence) is a cubic surface in the projective space $\mathbb{P}_{\mathbb{F}}^3$ over the field $\mathbb{F} = \mathbb{C}(\lambda)$, that is given by equation

$$(3.7) \quad \lambda^2 l_1 l_2 l_4 + \lambda q_3 + l_3 l_5 l_6 = 0.$$

Here we use x_0, \dots, x_3 also as a homogeneous coordinates on $\mathbb{P}_{\mathbb{F}}^3$.

Restricting the left hand side of equation (3.7) to the plane Π' given by $x_3 = x_0 + x_1 + x_2$ and using Remark 3.1, we see that the corresponding curve is given by equation

$$(3.8) \quad (\lambda l_4 + \sqrt{2\tau + 1}(2\tau - 3)l_3) (\lambda l_1 l_2 + \sqrt{2\tau + 1}l_5 l_6) = 0.$$

Restricting the left hand side of equation (3.7) to the plane Π'' given by $x_3 = x_0 - x_1 - x_2$ and using Remark 3.1, we see that the corresponding curve is given by equation

$$(3.9) \quad (\lambda l_1 + \sqrt{2\tau + 1}(2\tau - 3)l_6) (\lambda l_2 l_4 + \sqrt{2\tau + 1}l_3 l_5) = 0.$$

One can check that the lines in $\mathbb{P}_{\mathbb{F}}^3$ given by equations

$$x_3 - x_0 - x_1 - x_2 = \lambda l_4 + \sqrt{2\tau + 1}(2\tau - 3)l_3 = 0$$

and

$$x_3 - x_0 + x_1 + x_2 = \lambda l_1 + \sqrt{2\tau + 1}(2\tau - 3)l_6 = 0$$

are disjoint. Since they are contained in the cubic surface (3.7), we see that the cubic surface (3.7) is rational over the field \mathbb{F} , so that both Y and X are rational (over the field \mathbb{C}). \square

Now we are going to describe the generators of the group $\text{Cl}(X)$.

The intersection of X with the hypersurface $x_3 = x_0 + x_1 + x_2$ splits as a union of two surfaces $\Xi_{(1,1,1)}^+$ and $\Xi_{(1,1,1)}^-$ that are given by

$$\begin{cases} x_3 = x_0 + x_1 + x_2, \\ w = C_{\pm} \cdot \left((\tau - 2)(x_0 x_1^2 + x_1 x_2^2 + x_2 x_0^2) + (\tau - 3)x_0 x_1 x_2 - (x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0) \right), \end{cases}$$

respectively, where $C_{\pm} = \pm 2\sqrt{5\tau + 3}$. The image $\pi(\Xi_{(1,1,1)}^+) = \pi(\Xi_{(1,1,1)}^-)$ is the plane in \mathbb{P}^3 that is given by equation $x_3 = x_0 + x_1 + x_2$, cf. Lemma 3.2. Since the \mathfrak{A}_5 -orbit of the plane $x_3 = x_0 + x_1 + x_2$ consists of 20 planes, the \mathfrak{A}_5 -orbit of the surface $\Xi_{(1,1,1)}^+$ consists of 20 surfaces. Similarly, the \mathfrak{A}_5 -orbit of the surface $\Xi_{(1,1,1)}^-$ also consists of 20 surfaces. Denote by Ξ_v^+ the surface in the \mathfrak{A}_5 -orbit of $\Xi_{(1,1,1)}^+$ such that $\pi(\Xi_v^+) = \Xi_v$, where $v = (v_0, v_1, v_2)$ is one of the collections of coefficients listed in (3.5).

Remark 3.7. The anticanonical degree of the surfaces Ξ_v^{\pm} equals 1, i.e. one has $\Xi_v^{\pm} \cdot K_X^2 = 1$. By Lemma 3.2, the preimage on X of a plane Θ_u also splits as a union of two surfaces Θ_u^+ and Θ_u^- of anticanonical degree 1. It follows from Proposition 3.4 that there are no surfaces of anticanonical degree 1 on X except Ξ_v^{\pm} and Θ_u^{\pm} .

Fix a sufficiently general (smooth) $K3$ surface S in the linear system $|-K_X|$. For every two surfaces Ξ_v^+ and $\Xi_{v'}^+$, put

$$\Xi_v^+ \bullet \Xi_{v'}^+ = \Xi_v^+|_S \cdot \Xi_{v'}^+|_S.$$

Then $\Xi_v^+ \bullet \Xi_v^+ = -2$ by the adjunction formula. Moreover, if $v \neq v'$, then either $\Xi_v^+ \bullet \Xi_{v'}^+ = 1$ or $\Xi_v^+ \bullet \Xi_{v'}^+ = 0$ by construction. Furthermore, if $v \neq v'$, then $\Xi_v^+ \bullet \Xi_{v'}^+ = 0$ if and only if the intersection $\Xi_v^+ \cap \Xi_{v'}^+$ consists of finitely many points.

Denote by N , R , and M the transformations (3.2), (3.3), and (3.4), respectively.

Example 3.8. By Remark 3.1, the surface $\Xi_{(1,1,1)}^+$ is defined in $\mathbb{P}(1, 1, 1, 1, 2)$ by equations

$$\begin{cases} x_3 = x_0 + x_1 + x_2, \\ w = (\tau - 2)(x_0x_1^2 + x_1x_2^2 + x_2x_0^2) + (\tau - 3)x_0x_1x_2 - (x_0^2x_1 + x_1^2x_2 + x_2^2x_0), \end{cases}$$

The transformation M^3 is given by the matrix

$$M^3 = \frac{1}{2} \begin{pmatrix} 1 & \tau & 1 - \tau \\ \tau & 1 - \tau & 1 \\ \tau - 1 & -1 & -\tau \end{pmatrix}.$$

Thus the surface $\Xi_{(1,1,-1)}^+$ is defined by equations

$$\begin{cases} x_3 = x_0 + x_1 - x_2, \\ w = (\tau - 2)(x_0x_1^2 + x_1x_2^2 - x_2x_0^2) - (\tau - 3)x_0x_1x_2 - (x_0^2x_1 - x_1^2x_2 + x_2^2x_0). \end{cases}$$

Therefore, the intersection $\Xi_{(1,1,1)}^+ \cap \Xi_{(1,-1,-1)}^+$ is the line in $\Xi_{(1,1,1)}^+$ that is cut out by an equation $x_2 = 0$, so that $\Xi_{(1,1,1)}^+ \bullet \Xi_{(1,-1,-1)}^+ = 1$.

Example 3.9. The transformation RN is given by the matrix

$$RN = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Thus the surface $\Xi_{(1,-1,-1)}^+$ is defined in $\mathbb{P}(1, 1, 1, 1, 2)$ by equations

$$\begin{cases} x_3 = x_0 - x_1 - x_2, \\ w = (\tau - 2)(x_0x_1^2 - x_1x_2^2 - x_2x_0^2) + (\tau - 3)x_0x_1x_2 + (x_0^2x_1 + x_1^2x_2 - x_2^2x_0). \end{cases}$$

Therefore, the intersection $\Xi_{(1,1,1)}^+ \cap \Xi_{(1,-1,-1)}^+$ is defined by equations

$$\begin{cases} x_1 + x_2 = 0, \\ (\tau - 2)(x_1x_2^2 + x_2x_0^2) = x_0^2x_1 + x_1^2x_2. \end{cases}$$

This system of equations defines a set consisting of three points, and $\Xi_{(1,1,1)}^+ \bullet \Xi_{(1,-1,-1)}^+ = 0$.

In the similar way one gets $\Xi_v^+ = T_v(\Xi_{(1,1,1)}^+)$, where T_v is an element in \mathfrak{A}_5 given by Table 1.

TABLE 1. The surfaces Ξ_v^+

v	$(1, 1, 1)$	$(1, 1, -1)$	$(1, -1, 1)$	$(-1, 1, 1)$	$(1, -1, -1)$
T_v	Id	M^3	R^2M^3	RM^3	RN
v	$(-1, 1, -1)$	$(-1, -1, 1)$	$(-1, -1, -1)$	$(\tau - 1, \tau, 0)$	$(1 - \tau, \tau, 0)$
T_v	R^2N	N	M^2N	M^4	RM^2
v	$(1 - \tau, -\tau, 0)$	$(\tau - 1, -\tau, 0)$	$(\tau, 0, 1 - \tau)$	$(\tau, 0, \tau - 1)$	$(-\tau, 0, \tau - 1)$
T_v	MN	RM^4N	M^2	M	M^4N
v	$(-\tau, 0, 1 - \tau)$	$(0, \tau - 1, \tau)$	$(0, \tau - 1, -\tau)$	$(0, 1 - \tau, -\tau)$	$(0, 1 - \tau, \tau)$
T_v	M^3N	RM^4	R^2M^4N	RMN	R^2M^2

This leads to the following result.

Lemma 3.10. *One has*

$$\Xi_{(1,1,1)}^+ \bullet \Xi_v^+ = \begin{cases} 0 & \text{if } v = (1, -1, -1), (-1, 1, -1), (-1, -1, 1), (-1, -1, -1), \\ & (\tau - 1, -\tau, 0), (-\tau, 0, \tau - 1), (0, \tau - 1, -\tau), \\ 1 & \text{if } v = (1, 1, -1), (1, -1, 1), (-1, 1, 1), (\tau - 1, \tau, 0), (1 - \tau, \tau, 0), \\ & (1 - \tau, -\tau, 0), (\tau, 0, 1 - \tau), (\tau, 0, \tau - 1), (-\tau, 0, 1 - \tau), \\ & (0, \tau - 1, \tau), (0, 1 - \tau, -\tau), (0, 1 - \tau, \tau), \\ -2 & \text{if } v = (1, 1, 1). \end{cases}$$

Similarly, we can compute all possible values of $\Xi_v^+ \bullet \Xi_{v'}^+$. They are given in Table 2 below. The intersection matrix for the surfaces Ξ_v^- is the same as one given by Table 2.

Corollary 3.11. *The 20×20 matrix of intersection numbers of the surfaces Ξ_v^+ , where the index v is taken from the list (3.5), has rank 14. Similarly, the 20×20 matrix of intersection numbers of the surfaces Ξ_v^- has rank 14.*

Proof. Straightforward computation. □

Lemma 3.5 and Corollary 3.11 imply the following result.

Corollary 3.12. *The classes of the 20 surfaces Ξ_v^+ , where the index v is taken from the list (3.5), generate the \mathbb{Q} -vector space $\text{Cl}(X)_{\mathbb{Q}} = \text{Cl}(X) \otimes \mathbb{Q}$. Similarly, the classes of the 20 surfaces Ξ_v^- also generate the \mathbb{Q} -vector space $\text{Cl}(X)_{\mathbb{Q}}$.*

Corollaries 2.8 and 3.12 imply

Proposition 3.13. *The group $\text{Cl}(X)^G$ is generated by $-K_X$.*

Now we derive Theorem 1.2 from Proposition 3.13.

Proof of Theorem 1.2. Since $\text{Cl}(X)^G$ is generated by $-K_X$ by Proposition 3.13, the required assertion immediately follows from the proof of [CP10, Theorem A]. The only difference is that one should use a G -equivariant version of the standard Noether–Fano inequality, which is [CS15, Corollary 3.3.3]. □

TABLE 2. Intersection matrix for Ξ_v^+

	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,1)$	$(1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$	$(\tau-1,\tau,0)$	$(1-\tau,\tau,0)$	$(1-\tau,-\tau,0)$	$(\tau-1,-\tau,0)$	$(\tau,0,1-\tau)$	$(\tau,0,\tau-1)$	$(-\tau,0,\tau-1)$	$(-\tau,0,1-\tau)$	$(0,\tau-1,\tau)$	$(0,\tau-1,-\tau)$	$(0,1-\tau,-\tau)$	$(0,1-\tau,\tau)$
$(1,1,1)$	-2	1	1	1	0	0	0	0	1	1	1	0	1	1	0	1	1	0	1	1
$(1,1,-1)$	1	-2	0	0	1	1	0	0	1	1	1	0	1	1	1	0	0	1	1	1
$(1,-1,1)$	1	0	-2	0	1	0	1	0	0	1	1	1	1	1	0	1	1	1	0	1
$(-1,1,1)$	1	0	0	-2	0	1	1	0	1	1	0	1	1	0	1	1	1	0	1	1
$(1,-1,-1)$	0	1	1	0	-2	0	0	1	0	1	1	1	1	1	1	0	1	1	1	0
$(-1,1,-1)$	0	1	0	1	0	-2	0	1	1	1	0	1	0	1	1	1	0	1	1	1
$(-1,-1,1)$	0	0	1	1	0	0	-2	1	1	0	1	1	1	0	1	1	1	1	0	1
$(-1,-1,-1)$	0	0	0	0	1	1	1	-2	1	0	1	1	0	1	1	1	1	1	1	0
$(\tau-1,\tau,0)$	1	1	0	1	0	1	1	1	-2	1	0	1	1	1	0	0	1	1	0	0
$(1-\tau,\tau,0)$	1	1	1	1	1	1	0	0	1	-2	1	0	0	0	1	1	1	1	0	0
$(1-\tau,-\tau,0)$	1	1	1	0	1	0	1	1	0	1	-2	1	0	0	1	1	0	0	1	1
$(\tau-1,-\tau,0)$	0	0	1	1	1	1	1	1	1	0	1	-2	1	1	0	0	0	0	1	1
$(\tau,0,1-\tau)$	1	1	1	1	1	0	1	0	1	0	0	1	-2	1	0	1	0	1	1	0
$(\tau,0,\tau-1)$	1	1	1	0	1	1	0	1	1	0	0	1	1	-2	1	0	1	0	0	1
$(-\tau,0,\tau-1)$	0	1	0	1	1	1	1	1	0	1	1	0	0	1	-2	1	1	0	0	1
$(-\tau,0,1-\tau)$	1	0	1	1	0	1	1	1	0	1	1	0	1	0	1	-2	0	1	1	0
$(0,\tau-1,\tau)$	1	0	1	1	1	0	1	1	1	1	0	0	0	1	1	0	-2	1	0	1
$(0,\tau-1,-\tau)$	0	1	1	0	1	1	1	1	1	1	0	0	1	0	0	1	1	-2	1	0
$(0,1-\tau,-\tau)$	1	1	0	1	1	1	0	1	0	0	1	1	1	0	0	1	0	1	-2	1
$(0,1-\tau,\tau)$	1	1	1	1	0	1	1	0	0	0	1	1	0	1	1	0	1	0	1	-2

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